Chapter 3

Measurements

Sándor Imre
Ferenc Balázs
There are two possible outcomes: If the result confirms the hypothesis, then you’ve made a measurement. If the result is contrary to the hypothesis, then you’ve made a discovery.

Enrico Fermi
General Measurements

"WHY must I treat the measuring device classically?? What will happen to me if I don’t??"

Eugene Wigner
3rd Postulate using ket notations

- Measurement statistic
  \[ P(m \mid \varphi) = \langle \varphi \mid M_m^\dagger M_m \mid \varphi \rangle \]

- Post measurement state
  \[ |\varphi'\rangle = \frac{M_m |\varphi\rangle}{\sqrt{\langle \varphi \mid M_m^\dagger M_m |\varphi\rangle}} \]

- Completeness relation
  \[ \sum_m M_m^\dagger M_m \equiv I \]
Projective Measurements
(Neumann Measurements)

“Projective geometry has opened up for us with the greatest facility new territories in our science, and has rightly been called the royal road to our particular field of knowledge.”

Felix Klein
Measurement operators and the 3rd Postulate in case of projective measurements

• Set of two orthogonal states
  \[ |\varphi_0\rangle = |0\rangle \text{ or } |\varphi_1\rangle = |1\rangle \]

• To find \( M_0 \) we need to solve
  \[
  1 = \langle 0 | M_0^\dagger M_0 | 0 \rangle \\
  0 = \langle 1 | M_0^\dagger M_0 | 1 \rangle
  \]

• We are looking in the form of
  \[
  M_0 = \begin{bmatrix}
  a & b \\
  c & d
  \end{bmatrix}
  \]

  \[
  1 = |a|^2 + |c|^2 \\
  0 = |b|^2 + |d|^2
  \]

• Similarly
  \[
  M_1 = \begin{bmatrix}
  0 & 0 \\
  0 & 1
  \end{bmatrix}
  \]
Measurement operators and the 3rd Postulate
in case of projective measurements

- Checking the Completeness relation

\[ \sum_m M_m^\dagger M_m = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \]

- Practical notation

\[ M_0 = |0\rangle\langle 0| \text{ and } M_1 = |1\rangle\langle 1| \]

- Conclusion

Thus we reached a very simple and practical rule of thumb: In case we have a set of orthonormal states \( \{|\varphi_m\rangle\} \) then the corresponding measurement operators which provide exact differentiation among them can be produced by \( M_m = |\varphi_m\rangle\langle \varphi_m| \).
Measurement operators and the 3rd Postulate in case of projective measurements

- $M_m$ belong to a special set of operators called projectors $P_m$

- Properties

1. Obviously they are self-adjoint operators $P_m^\dagger \equiv P_m$ since $(|\varphi_m\rangle\langle \varphi_m|)^\dagger = <\varphi_m|\varphi_m> = |\varphi_m\rangle\langle \varphi_m|.$

2. Furthermore $P_mP_m = |\varphi_m\rangle\langle \varphi_m| = P_m.$

3. Finally they are orthogonal which means $P_mP_n = |\varphi_m\rangle\langle \varphi_m|\langle \varphi_n| = \delta(m - n)P_m.$
Measurement operators and the 3rd Postulate in case of projective measurements

• 3rd Postulate with projectors

\[ P(m \mid \varphi) = \langle \varphi | P_m | \varphi \rangle \]

\[ |\varphi'\rangle = \frac{P_m |\varphi\rangle}{\sqrt{\langle \varphi | P_m | \varphi \rangle}} \]

\[ \sum_m P_m \equiv I \]

Exercise 3.1. Construct the measurement operators providing sure success in case of the following set \( |\varphi_0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \) and \( |\varphi_1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \)!
Measurement operators and the 3rd Postulate in case of projective measurements

• **Direct construction approach**
  In case of *direct approach* we start from (3.8) and we would like to ensure $P(m \mid \varphi_m) = 1$ if $\varphi_m$ is received. This can be done using parallel vectors to $\varphi_m$ in the outer product representation of $P_m$ thus $P(m \mid \varphi_m) = \langle \varphi_m \mid \varphi_m \rangle \langle \varphi_m \mid \varphi_m \rangle = 1$. Using the same technique for all possible $m$ we form the set of measurement operators.

• **Indirect construction approach**
  According to the *indirect approach* we start from (3.8) again but we would like to ensure $P(n \mid \varphi_m) = 0$, $\forall n \neq m$ if $\varphi_m$ is received. This can be achieved applying orthogonal vectors to $\varphi_m$ in the outer product representation of $P_m$ thus $P(n \mid \varphi_m) = \langle \varphi_m \mid \varphi_n \rangle \langle \varphi_n \mid \varphi_m \rangle = 0$, $\forall n \neq m$ and because $\sum_l P(l \mid \varphi_m) = 1$ hence $P(m \mid \varphi_m) = 1$. Using the same technique for all possible $m$ we form the set of measurement operators.
Measurement using the computational basis states

- Let us check what we have learned by means of a simple example
  \[ |\varphi\rangle = a|0\rangle + b|1\rangle \]
- Basis vectors \(|0\rangle\) and \(|1\rangle\)

\[
P(0 \mid |\varphi\rangle) = \langle \varphi | P_0 | \varphi \rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \]

\[
P(1 \mid |\varphi\rangle) = \langle \varphi | P_0 | \varphi \rangle = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \]
Measurement using the computational basis states

- Post measurements states

\[
|\varphi_0\rangle = \frac{P_0 |0\rangle}{\sqrt{P(0 | \varphi\rangle)}} = \frac{a |0\rangle}{|a|}
\]

\[
|\varphi_1\rangle = \frac{P_1 |0\rangle}{\sqrt{P(1 | \varphi\rangle)}} = \frac{b |1\rangle}{|b|}
\]

- **Remark:** Orthogonal states can always be distinguished via constructing appropriate measurement operators (projectors). This is another explanation why orthogonal (classical) states can be copied as was stated in Section 2.7 because in possession of the exact information about such states we can build a quantum circuit producing them.
Observable and projective measurements

• Any observable can be represented by means of a Hermitian operator whose eigenvalues refer to the possible values of that observable

\[ K = \sum_m mP_m \]

• Expected value of such an observable can be calculated in an easy way

\[
\mathbb{E}(K) = \sum_m mP(m \mid \varphi) = \sum_m m\langle \varphi \mid P_m \mid \varphi \rangle \\
\quad = \langle \varphi \mid \left( \sum_m mP_m \right) \mid \varphi \rangle = \langle \varphi \mid K \mid \varphi \rangle
\]
Repeated projective measurement

- What happens when we repeat a projective measurement on the same qregister?
- Post measurement state after the first measurement

\[ |\varphi_m\rangle = \frac{P_m |\varphi\rangle}{\sqrt{\langle \varphi | P_m | \varphi \rangle}} \]

- Post measurement state after the second measurement

\[ P_m = |\varphi_m\rangle \langle \varphi_m| \]

\[ |\varphi_m\rangle' = \frac{P_m |\varphi_m\rangle}{\sqrt{\langle \varphi_m | P_m | \varphi_m \rangle}} = \frac{|\varphi_m\rangle \langle \varphi_m| |\varphi_m\rangle}{\sqrt{\langle \varphi_m | P_m | \varphi_m \rangle}} = |\varphi_m\rangle \]
CHSH inequality with entangled particles

- We return to Bell inequalities and investigate CHSH inequality in the nano-world. We replace books with entangled pairs

\[ |\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \]

- Alice and Bob check the following observables

\[
\begin{align*}
O_A &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
O_B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
O_C &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \\
O_D &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}
\end{align*}
\]
CHSH inequality with entangled particles

\[ \mathbb{E}(s) = \mathbb{E}(AC') + \mathbb{E}(AD) + \mathbb{E}(BC') - \mathbb{E}(BD) \]

\[ \mathbb{E}(s) = \langle \beta_{11} | O_{AC} | \beta_{11} \rangle + \langle \beta_{11} | O_{AD} | \beta_{11} \rangle + \langle \beta_{11} | O_{BC} | \beta_{11} \rangle + \langle \beta_{11} | O_{BD} | \beta_{11} \rangle \]

\[ \mathbb{E}(AC') \triangleq \sum_{\alpha} \sum_{\gamma} \alpha \gamma P(A = \alpha \land C = \gamma | \psi) = \sum_{\alpha} \sum_{\gamma} \alpha \gamma \langle \psi | P_{\alpha,\gamma} | \psi \rangle \]

\[ = \langle \psi | \sum_{\alpha} \sum_{\gamma} \alpha \gamma P_{\alpha,\gamma} | \psi \rangle. \]

\[ P_{\alpha,\gamma} = P_{\alpha} \otimes P_{\gamma}. \]

\[ \mathbb{E}(AC') = \langle \psi | \sum_{\alpha} \sum_{\gamma} \alpha \gamma (P_{\alpha} \otimes P_{\gamma}) | \psi \rangle = \langle \psi | \left( \sum_{\alpha} \alpha P_{\alpha} \right) \otimes \left( \sum_{\gamma} \gamma P_{\gamma} \right) | \psi \rangle \]

\[ = \langle \psi | O_A \otimes O_C | \psi \rangle. \]
CHSH inequality with entangled particles

\[ O_A = \sum_{\alpha} \alpha P_\alpha, \quad E(A) = \langle \varphi_1 | O_A | \varphi_1 \rangle, \]

\[ O_C = \sum_{\gamma} \gamma P_\gamma, \quad E(C) = \langle \varphi_2 | O_C | \varphi_2 \rangle. \]

• Finally we obtain \( E(s) = 2\sqrt{2} \) which is obviously greater than the classical result 2.
• Experiments shore up the quantum model instead of the classical one!
• Hidden variables do not exist!
Positive Operator Valued Measurements

Good quotation is requested. If you have one please, send to imre@hit.bme.hu
Motivations to use POVM

• In certain cases either we are not interested in the post measurement state or we are not able at all to get it (e.g. a photon hits the detector).

\[ M_m^\dagger M_m \rightarrow D_m \]

• Important properties of \( D_m \)

1. Obviously they are self-adjoint operators because \( D_m^\dagger = (M_m^\dagger M_m)^\dagger = M_m^\dagger M_m = D_m \).

2. Any operator in the form of \( |\varphi_m\rangle\langle\varphi_m| \) is positive semi-definite since for all \( |\psi\rangle \), \( P(m \mid |\psi\rangle) = \langle\psi|\varphi_m\rangle\langle\varphi_m|\psi\rangle = \langle\psi|\varphi_m\rangle(\langle\psi|\varphi_m\rangle)^* = |\langle\psi|\varphi_m\rangle|^2 \geq 0 \). Because both \( |\psi\rangle \) and \( |\varphi_m\rangle \) are unit length vectors thus \( P(m \mid |\psi\rangle) \leq 1 \).

3. This latter statement is also true from the opposite direction because \( D_m \) is constructed in the form of \( M_m^\dagger M_m \).
POVM and the 3rd Postulate

• Measurement statistic

\[ P(m \mid \varphi) = \langle \varphi \mid D_m \mid \varphi \rangle \]

• Post measurement state

Unknown and/or indifferent!

• Completeness relation

\[ \sum_m D_m = I \]
Can non-orthogonal states be distinguished?

• The answer is definitely NOT because

\[ P(m \mid \varphi_n) = \langle \varphi_n \mid D_m \mid \varphi_n \rangle = (\langle \varphi_m' \mid + \langle \varphi_m'' \mid | \varphi_m \rangle \langle \varphi_m' \mid (| \varphi_m'' \rangle + | \varphi_m'\rangle) \]

\[ = \langle \varphi_m'' \mid \varphi_m \rangle \langle \varphi_m \mid \varphi_m'' \rangle + \langle \varphi_m'' \mid \varphi_m \rangle \langle \varphi_m \mid \varphi_m'' \rangle \]

\[ \equiv 0 \quad \equiv 0 \quad \neq 0 \quad \neq 0 \]

\[ + \langle \varphi_m'' \mid \varphi_m \rangle \langle \varphi_m \mid \varphi_m'' \rangle + \langle \varphi_m'' \mid \varphi_m \rangle \langle \varphi_m \mid \varphi_m'' \rangle \neq 0. \]

\[ \equiv 0 \quad \neq 0 \quad \neq 0 \quad \equiv 0 \]

• **Remark:** No set of measurement operators exist which is able to distinguish non-orthogonal states unambiguously. This is another explanation why non-orthogonal states cannot be copied as was stated earlier, because of lack of exact information about such states we cannot build a quantum circuit to produce them.
POVM construction example

- Our two non-orthogonal states are:
  \[ |\varphi_0\rangle = |0\rangle, \quad |\varphi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad \langle \varphi_0 | \varphi_1 \rangle = \frac{1}{\sqrt{2}} \neq 0. \]
- Based on the indirect construction approach we try to ensure that
  \[ P(1 \mid |\varphi_0\rangle) = \langle \varphi_0 | D_1 | \varphi_0 \rangle = 0, \quad P(0 \mid |\varphi_1\rangle) = \langle \varphi_1 | D_0 | \varphi_1 \rangle = 0. \]
- To achieve this we need \( \alpha, \beta \in \mathbb{C} \)

  \[
  D_0 = \alpha \frac{|0\rangle - |1\rangle}{\sqrt{2}} \frac{\langle 0| - \langle 1|}{\sqrt{2}}, \quad D_1 = \beta |1\rangle \langle 1| \\
  D_0 = \begin{bmatrix} \frac{\alpha}{2} & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & \frac{\alpha}{2} \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}
  \]
POVM construction example

• To fulfil the completeness relation

\[ D_2 = I - D_0 - D_1 \Rightarrow D_2 = \begin{bmatrix} 1 - \frac{\alpha}{2} & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 1 - \beta - \frac{\alpha}{2} \end{bmatrix} \]
\[ P(m \mid |\varphi_k\rangle) = \langle \varphi_k \mid D_m \mid \varphi_k \rangle \]

\[ P(0 \mid |\varphi_0\rangle) = \begin{bmatrix} \frac{\alpha}{2} & -\frac{\alpha}{2} \\ \frac{\alpha}{2} & \frac{\alpha}{2} \end{bmatrix}, \quad P(1 \mid |\varphi_0\rangle) = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}, \quad P(2 \mid |\varphi_0\rangle) = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{\alpha}{2} \end{bmatrix} \]

\[ P(1 \mid |\varphi_1\rangle) = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}, \quad P(2 \mid |\varphi_1\rangle) = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{\alpha}{2} \end{bmatrix} \]

- Completeness relation OK since

\[ \sum_{m=0}^{2} P(m \mid |\varphi_0\rangle) = 1, \quad \sum_{m=0}^{2} P(m \mid |\varphi_1\rangle) = 1 \]
\(D_2\) must be positive semi-definite

- \(\alpha = \beta = 2\) is very promising, but unfortunately wrong!

\[
D_2 = |\varphi_2\rangle\langle \varphi_2| = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} 1 - \frac{\alpha}{2} & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 1 - \beta - \frac{\alpha}{2} \end{bmatrix}
\]

from which

\[
a = \sqrt{1 - \frac{\alpha}{2}}, \quad b = \sqrt{1 - \beta - \frac{\alpha}{2}},
\]

and we have the following constraint

\[
\frac{\alpha}{2} = \sqrt{1 - \frac{\alpha}{2}} \sqrt{1 - \beta - \frac{\alpha}{2}} \implies \beta = \frac{1 - \alpha}{1 - \frac{\alpha}{2}}.
\]
How to apply POVM operators - 0 and 1 has the same importance

- It requires

\[ P(2 | \varphi_0) = P(2 | \varphi_1) \]

\[ \alpha = \beta = 2 - \sqrt{2}. \]
How to apply POVM operators - minimising uncertainty

• Goal: to minimise $P(m = 2)$

$$P(m = 2) = P(2 \mid \varphi_0)P(\varphi_0) + P(2 \mid \varphi_1)P(\varphi_1)$$

• Assuming $P(\varphi_k) = \frac{1}{2}$

$$P(m = 2) = -\frac{1}{4} \frac{\alpha^2 - 4\alpha + 6}{\alpha - 2}$$

$$\frac{dP(m = 2)}{d\alpha} = -\frac{1}{4} \frac{\alpha^2 - 4\alpha + 2}{(\alpha - 2)^2} = 0$$

$$\alpha = \beta = 2 - \sqrt{2}.$$
How to apply POVM operators - minimising uncertainty
How to apply POVM operators - false alarm vs. not happen alarm

- It is assumed that detecting 1 correctly is much more important

\[
P(1 \mid \varphi_1) = 0.5.
\]
\[
P(0 \mid \varphi_0) = 0
\]
Generalisation of POVM

Finally we emphasize a two-step generalization of the above example. First if we use a set of $n$-bit long linearly independent states $\{ |\varphi_k\rangle \}$ with $N$ elements ($k = 0, 1\ldots N - 1$) where obviously $N \leq 2^n - 1$, then in compliance with the above explained technique one can form corresponding POVM operators $D_m, m = 0, 1\ldots N$ which enables correct answers if the measuring equipment indicates $m < N$ and we become indecisive only in case of $m = N$. 
Relations among the measurement types

- Projective measurement can be regarded as a special POVM.
- Clearly speaking POVM is a generalised measurement without the interest of post-measurement state + construction rules.
- Neumark’s extension: any generalised measurement can be implemented by means of a projective measurement + auxiliary qbits + unitary transform.
Relations among the measurement types
Quantum computing-based solution of the game with marbles

"Victorious warriors win first and then go to war, while defeated warriors go to war first and then seek to win.” Sun Tzu

- We exploit entanglement and the orthogonality between Bell states

\[
|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\beta_{01}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \quad \mathbf{P}(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\alpha} \end{bmatrix} \quad \alpha = \pi
\]

\[
(H \otimes I)|\beta_{00}\rangle = |\beta_{10}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}
\]

\[
(H \otimes I)|\beta_{01}\rangle = |\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}
\]
Quantum computing-based solution of the game with marbles

| 00⟩ + |11⟩ \over \sqrt{2} |